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An analytic solution describing the motion of a bore over a sloping beach

By NEELAM GUPTA
Ramrao Adik Institute of Technology, Dr D. Y. Patil Vidyanagar, Nerul, New Bombay 400706, India

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An exact analytic solution of the shallow water equations for the motion of a bore over a uniformly sloping beach is derived. This solution is valid only when the initial bore is supercritical. The results agree very well with those of Keller, Levine & Whitham (1960).

1. Introduction

The problem of a bore travelling shoreward into water at rest on a sloping beach was first studied as an application of Whitham's characteristic rule (Whitham 1958). This rule, in effect, assumes that a differential relation, holding strictly along a positive characteristic, is applicable to the flow quantities immediately behind the bore and provides an approximate formula for the variation in the strength and height of the bore. The results of the approximate rule agreed well with the numerical solution of Keller, Levine & Whitham and the behaviour of the bore near the shoreline was found to be completely independent of the initial motion producing the bore.

The characteristic rule was connected with the modulated simple wave theory of Varley, Ventakaraman & Cumberbatch (1971) by Sachdev & Seshadri (1976). Specifically, to derive an approximate solution the flow in the entire region behind the bore was assumed to be a modulated simple wave.

In the present paper we adopt an exact analytical approach. To obtain an explicit solution we consider the shallow water equations in the semi-characteristic plane (α, β), such that the level lines $\alpha =$ constant have slope equal to the particle velocity u, and $\beta =$ constant demarcates a negative characteristic. The legitimacy of this co-ordinate system is not obvious. However, since the analytic solution to be sought here involves a sufficient number of arbitrary functions, the propagation of the bore is described accurately up to the time for it to reach the shoreline.

2. Formulation

We consider the propagation of a bore on a beach of constant slope h_0' for $0 < x \leqslant x_0$. If $g^{-1}c^2$ is the depth of the water (where c is sound speed) and $u(x, t)$ is the particle velocity, the equations of shallow water theory are

$$u_t + uu_x + 2cc_x = H_0', \tag{2.1}$$

$$c_t + uc_x + \tfrac{1}{2}cu_x = 0. \tag{2.2}$$

Here, the prime denotes differentiation with respect to x. The bore conditions are

$$(u-U)\,c^2 + H_0\,U = 0, \tag{2.3}$$

$$c^2(c^2+H_0) - 2H_0\,U^2 = 0, \tag{2.4}$$

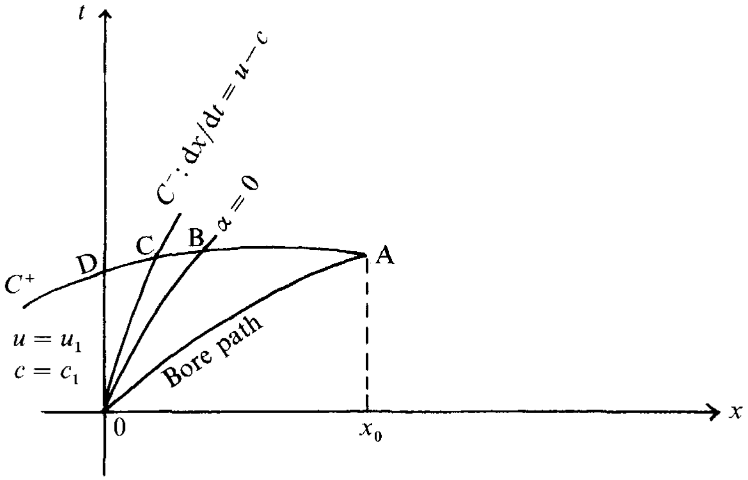

FIGURE 1. The (x, t)-diagram for $u_1 > c_1$.

where U is the bore velocity. In writing (2.1)–(2.4), u, c and U have been normalized by $(gh(0))^{\frac{1}{2}}$, the distance x by x_0 and time t by $x_0(gh(0))^{-\frac{1}{2}}$.

We assume that

$$H_0 = h_0(x)/h_0(0) = \begin{cases} 1, & x < 0, \\ 1-x, & 0 \leqslant x \leqslant 1, \end{cases} \tag{2.5}$$

and that the bore moves with a constant speed in the region $x < 0$, and arrives at $x = 0$ at time $t = 0$.

For the case when the bore is initially supercritical, i.e. constant values behind the bore are such that $u_1 > c_1$, the (x, t) diagram is shown in figure 1. Here, the flow is undisturbed with $u = u_1$ and $c = c_1$ in the region $x \leqslant 0$.

We introduce the variables α and β according to

$$\alpha_t + u\alpha_x = 0, \tag{2.6}$$

$$\beta_t + (u-c)\beta_x = 0, \tag{2.7}$$

as the new independent variables. This implies that β remains constant along a negative characteristic; α = constant is not a characteristic curve of equations (2.1) and (2.2). In consequence of this (2.1) and (2.2) may be transformed as

$$u_\alpha t_\beta - (u_\beta + 2c_\beta)\, t_\alpha = 0, \tag{2.8}$$

$$(u_\beta + 2c_\beta)\, t_\alpha + (t_\alpha - 2c_\alpha)\, t_\beta = 0. \tag{2.9}$$

The variables x and t are now dependent variables, which from (2.6) and (2.7), satisfy

$$x_\alpha = (u-c)\, t_\alpha, \tag{2.10}$$

$$x_\beta = ut_\beta. \tag{2.11}$$

Cross-differentiation yields

$$ct_{\alpha\beta} + 3t_\alpha c_\beta = 0, \tag{2.12}$$

which integrates to give

$$t_\alpha = c^{-3}F(\alpha), \tag{2.13}$$

where $F(\alpha)$ is an arbitrary function of α. We may note from (2.8) and (2.9) that

$$u-2c+t = G(\beta), \tag{2.14}$$

where $G(\beta)$ is a function of its argument. On using (2.14) and assuming that

$$c = c(\beta), \quad F = 1, \tag{2.15}$$

we observe that (2.9)–(2.11) and (2.13) admit an exact solution:

$$u(\alpha,\beta) = \tfrac{1}{2}(G+K)-t, \tag{2.16}$$

$$t(\alpha,\beta) = I(\beta)+c^{-3}\alpha, \tag{2.17}$$

$$c(\beta) = \tfrac{1}{4}(K-G), \tag{2.18}$$

$$x(\alpha,\beta) = (u-c)(t-I)+\tfrac{1}{2}(t-I)^2+Q(\beta), \tag{2.19}$$

where $Q(\beta)$ satisfies the relation

$$\frac{\mathrm{d}Q}{\mathrm{d}\beta} = [\tfrac{1}{2}(G+K)-I]\frac{\mathrm{d}I}{\mathrm{d}\beta}. \tag{2.20}$$

Here, the choice of c and F as made in (2.15) does not change the behaviour of flow variables behind the bore, for α varies very slowly along the negative characteristic $\beta = \text{constant}$. We may choose $G = \beta$, since the independent variable β does not appear explicitly in (2.16)–(2.20). Solution (2.16)–(2.19) now involves only one arbitrary function, namely $I(\beta)$. To obtain the solution in the entire region behind the bore, we rewrite u, c, t and α as functions of x and β. In the region OCD the solution is given by

$$\left.\begin{aligned} &u = u_1-t, \quad c = c_1, \quad t = c_1^{-3}\alpha, \\ &x = (u_1-c_1)(t-I_1)+\tfrac{1}{2}(t-I_1)^2+Q_1, \end{aligned}\right\} \tag{2.21}$$

where $I_1 = I(\beta_1), Q_1 = Q(\beta_1)$ with $\beta = \beta_1$ as the first negative characteristic originating from $x = t = 0$. If $I_1 = 0$ and $Q_1 = 0$, this satisfies the boundary conditions exactly on the line OD separating the uniform region.

The above solution is valid at least until the formation of a second bore. We notice that the characteristics which originate in region $x < 0$ have the same slope in $x > 0$, while the characteristics emanating in $x > 0$ fan out downstream more than those from $x < 0$. This shows that the negative characteristics do not intersect anywhere in the region $x > 0$. The second bore, therefore, is not a reflection of the original bore, but a hydraulic discontinuity formed by backwater down the beach.

Equation (2.19) when evaluated at the bore, yields the bore path

$$x_b = (u-c)(t_b-I)+\tfrac{1}{2}(t_b-I)^2+Q, \tag{2.22}$$

where x_b and t_b are the coordinates of the bore. Equations (2.16)–(2.20) also yield that

$$M = UH^{-\frac{1}{2}} \propto H_0^{-\frac{1}{4}}, \quad H-H_0 \propto H_0^{\frac{3}{2}} \tag{2.23}$$

as $H_0 \to 0$ when M is large. This asymptotic result has been noted earlier by Keller *et al.*

3. Discussion

For computation, we have chosen

$$\alpha = 0, \quad I = 0, \quad Q = 0, \quad G = \beta = u_1-2c_1, \quad K = u_1+2c_1$$

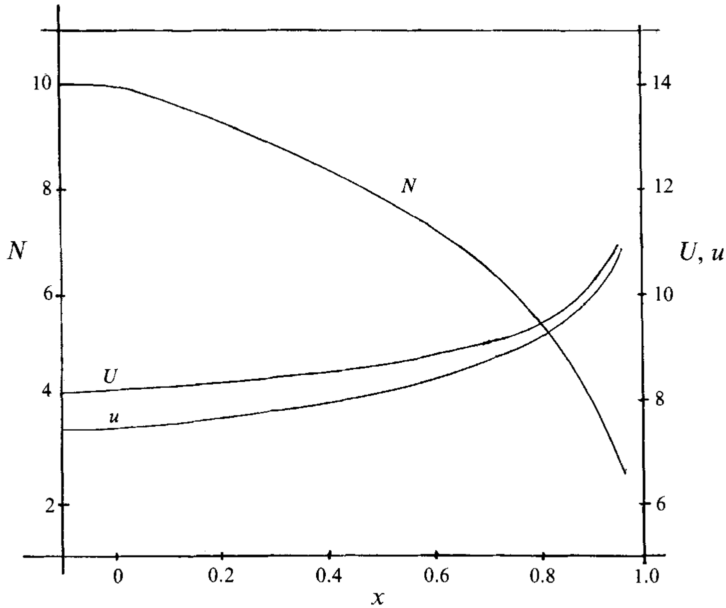

FIGURE 2. Variation of the bore height N, bore velocity U and particle velocity u in the case $N(0) = 10$.

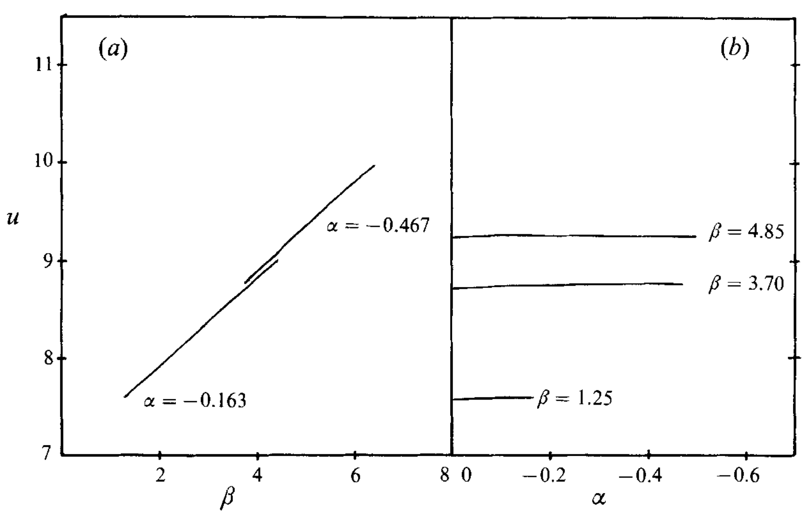

FIGURE 3. Particle velocity versus (a) β on lines α = constant, and (b) α on lines β = constant for $N(0) = 10$.

at $x = t = 0$. Figure 2 shows the results for the initial bore velocity $U = 8.12$. The corresponding initial flow behind the bore is supercritical. In this case bore height $N(= c^2 - H_0)$ decreases and U increases all the way to the shoreline. Moreover, as $H_0 \to 0$, u and U approach the same limiting value. The results are in such close agreement with the numerical solution (figure 3*b* of Keller *et al.*) that they are quite indistinguishable. Figure 3 shows the variation of u along α = constant and β = constant, while that of c is shown in figure 4. Since c remains constant along β =

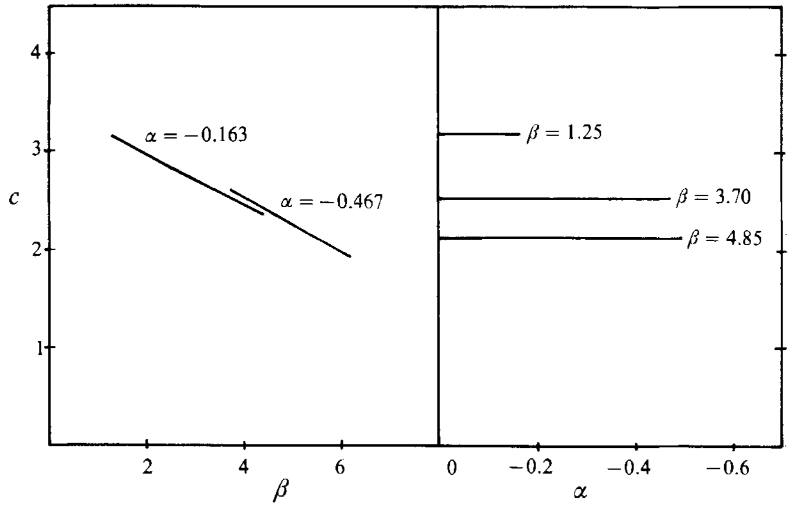

FIGURE 4. Sound speed versus (a) β on lines α = constant, and (b) α on lines β = constant for $N(0) = 10$.

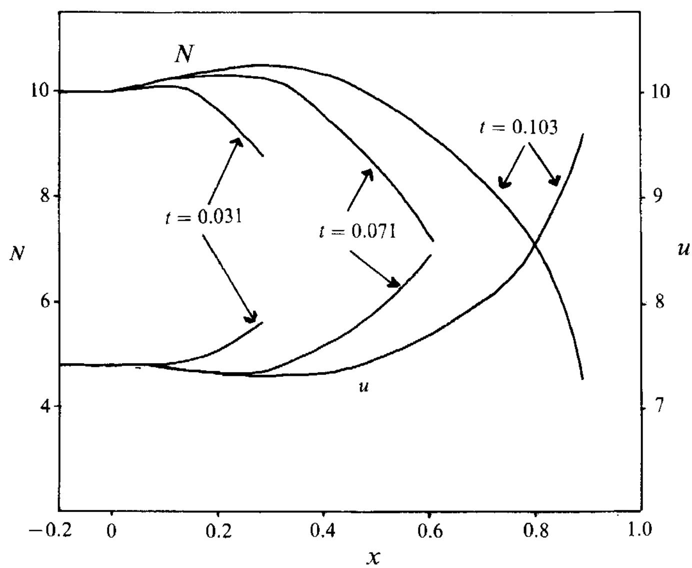

FIGURE 5. The flow behind the bore: height N and particle velocity u for $N(0) = 10$.

constant, a very small variation in u makes the negative Riemann invariant virtually constant along a negative-going characteristic C^-: $dx/dt = u - c$, which in the physical plane represents waves reflected from the bore, and monotonically increasing with the bore strength; the value of β increases as the bore strength increases. This result is consistent with that for flows headed by a shock (see figure 1 of Sirovich & Chong 1980, and figure 4 of Chong & Sirovich 1980). The flow behind the bore as described by the present solution is shown in figure 5. Again, graphical comparison between the analytic and the numerical solution (figure 4 b of Keller *et al.*) is not possible, since they are practically indiscernible. The prediction of the absence of a second bore as a reflection of the original one is in consonance with the absence of a subsidiary shock in regions of expanding area changes (Friedman 1960). We have not compared our results with those of Sachdev & Seshadri because they are approximate. Unfortunately, if $u_1 < c_1$, the solution (2.16)–(2.19) breaks down too soon. For the case when $N(0) = 0.25$, for example, the solution at the bore matches that calculated by Keller *et al.* only for $0 < x \leqslant 0.23$. It happens in this case because I increases and becomes equal to $\frac{1}{2}(G+K)$ at $x = 0.23$; I attains a maximum value at $x = 0.56$, making u zero and t decrease thereafter.

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